

Emergent 3-manifolds from 4d Superconformal Indices

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We show that the smooth geometry of a hyperbolic 3-manifold emerges from a classical spin system defined on a 2d discrete lattice, and moreover show that the process of this “dimensional oxidation” is equivalent with the dimensional reduction of a supersymmetric gauge theory from 4d to 3d. More concretely, we propose an equality between (1) the 4d superconformal index of a 4d $\mathcal{N} = 1$ superconformal quiver gauge theory described by a bipartite graph on T^2 and (2) the partition function of a classical integrable spin chain on T^2 . The 2d spin system is lifted to a hyperbolic 3-manifold after the dimensional reduction and the Higgsing of the 4d gauge theory.

Introduction.— The concept of spacetime has been of crucial importance in our understanding of Nature. However, in the theory of quantum gravity, it is widely believed that even the notion of classical spacetime is of secondary nature, and emerges from a more fundamental structure. One proposal for such a structure is the spin network [1], a spin system defined on a discrete lattice.

In a different line of development, more recently there have been important developments in supersymmetric gauge theories suggesting that the spacetime geometry could be traded for another “internal” geometry. This has been discussed for a class of supersymmetric gauge theories compactified on a compact curved manifold, which is thought of as the Euclidean version of the spacetime for the theory. The idea is simple; we begin with a D -dimensional field theory and compactify the theory on a class of d_1 -dimensional manifolds \mathcal{C} . The resulting d_2 -dimensional theory is defined on a fixed d_2 -dimensional compact manifold \mathcal{S} , where $d_1 + d_2 = D$. We could instead first compactify on \mathcal{S} , and then we have a d_1 -dimensional theory on \mathcal{C} . Thus we have a correspondence between the d_2 -dimensional field theory on \mathcal{S} and the d_1 -dimensional field theory on \mathcal{C} .

While the idea itself is rather general, in practice it is a rather difficult problem to make a precise identification between the observables of the two theories, since a quantity on one side could take a rather different form on the other. A successful example of such a quantitative identification is the relation between the S^4 partition function of 4d $\mathcal{N} = 2$ superconformal field theories (SCFT) arising from a compactification of 6d $(2, 0)$ theory on a Riemann surface C [2] and a correlation function of 2d Liouville theory on C [3].

The goal of this Letter is to unify these two apparently unrelated ideas in supersymmetric gauge theories and gravity. This gives new perspectives on the emergence of classical geometry, and surprisingly the process has a counterpart in the supersymmetric gauge theory.

We analyze the 4d superconformal index for quiver gauge theories dual to toric Calabi-Yau 3-folds, and find that the 4d index is equivalent to the partition function of an integrable spin system in 2d. We then discuss dimensional reduction from the 4d index to the 3d partition

function of the supersymmetric gauge theory; this is to take a particular limit of the 4d index. Surprisingly, on the 2d spin system side this limit is translated into a limit where classical/quantum geometry of a hyperbolic 3-manifold [21] emerges from the 2d lattice. In other words, emergent geometry, which is nothing but a dimensional oxidation [22] in our context, is translated into a dimensional reduction in the other description! What is novel about this story is that the emergent geometry on one side, where fluctuations of the geometry are described by quantum hyperbolic geometry, is translated into a simple dimensional reduction on the other side, with a fixed background (compactification manifold); the fluctuation of the background is traded for the fluctuation of the gauge theory degrees of freedom on a fixed background.

We expect that this is a general feature of the correspondence between d_1 and d_2 -dimensional theories mentioned above. Since we are splitting the D -dimensions into two, when we dimensionally *reduce* the theory on the d_2 -dimensional side, the dimension *increases* on the d_1 -dimensional side. The theories in this Letter provide a concrete example of this phenomenon.

Further details will be presented in a separate publication [4].

4d versus 2d.— We begin with a 4d $\mathcal{N} = 1$ quiver superconformal field theory obtained by probing toric Calabi-Yau 3-fold by N D3-branes. In the following we take $N = 2$. Here a quiver gauge theory is a gauge theory defined from an oriented graph (quiver); a vertex represents the $SU(2)$ gauge group and an edge represents a bifundamental matter field. In our case, the quiver is described by a set of zig-zag paths on T^2 (see Figure 1) [5]. The paths represent the primitive normals of the toric diagram and divide T^2 into the polygonal regions, each of which is colored black/white if all the paths around the region have counterclockwise/clockwise orientation and is uncolored otherwise. This determines the quiver diagram \mathcal{G} or its dual \mathcal{G}^* , written on T^2 . We denote the set of edges/faces/vertices of \mathcal{G} by E, F, V , E/F is the same as the colored/uncolored regions, and \mathcal{G} is a bipartite graph [6]. In gauge theory language V is the $SU(2)$ gauge group, E bifundamental matter and F the

superpotential term. We denote the endpoints of an edge $e \in E$ by $s(e), t(e)$. In short, zig-zag paths determine the UV Lagrangian of our theory.

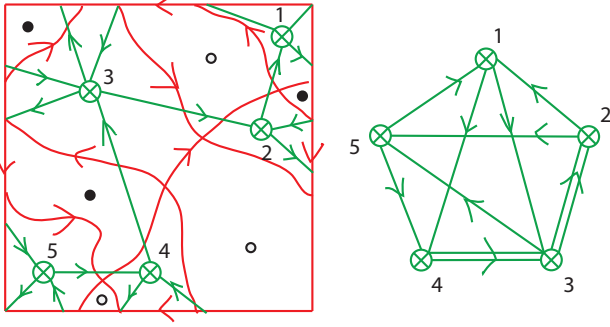


FIG. 1: On a two-dimensional torus \mathbb{T}^2 (the square shown on the left figure) we have a set of zig-zag paths, which divide the torus into black/white regions represented by black/white vertices and uncolored regions represented by green vertices. The right figure shows the corresponding quiver diagram.

There is an ambiguity in the choice of zig-zag paths, but assuming minimality all possible choices are related by the two moves [7], representing Seiberg duality and integrating out massive bifundamental matters. These preserve the IR fixed point.

Given a 4d $\mathcal{N} = 1$ SCFT we can define the superconformal index [8] (defined on $S^3 \times S^1$) by

$$I(p, q) = \text{Tr} \left[(-1)^F p^{\frac{\mathcal{E}+j_2}{3}+j_1} q^{\frac{\mathcal{E}+j_2}{3}-j_1} \right], \quad (1)$$

where the index is taken over the Hilbert space on S^3 , and F, \mathcal{E}, j_1, j_2 are the fermion number, the energy, and the spins under $SO(4) = SU(2) \times SU(2)$ rotation symmetry [23]. This is essentially the famous Witten index, except that we here include the chemical potentials for all the possible operators commuting with the supercharge. Since the index is independent of the parameters of the Lagrangian [8], we can compute the index in the free field limit and we have

$$I = \int_{|z_v|=1} \prod_{v \in V} \frac{dz_v}{z_v} \prod_{v \in V} I_{\text{vect}}^v(z) \prod_{e \in E} I_{\text{hyper}}^e(z; R), \quad (2)$$

where the integral is over the Cartan $(z_v, z_v^{-1}) \in U(1)_v \subset SU(2)_v$ of the gauge group at vertex v . The contribution from a gauge group at $v \in V$ is

$$I_{\text{vect}}^v(z) = \frac{\kappa}{2} \prod_{\epsilon=\pm 1} \Gamma(z_v^{2\epsilon}; p, q)^{-1}, \quad (3)$$

and that from a bifundamental at $e \in E$ is

$$I_{\text{hyper}}^e(z; R) = \prod_{\epsilon_1, \epsilon_2=\pm 1} \Gamma((pq)^{\frac{R_e}{2}} z_{s(e)}^{\epsilon_1} z_{t(e)}^{\epsilon_2}; p, q), \quad (4)$$

where R_e is the R-charge for the bifundamental, satisfying the conditions that the superpotential has R-charge

2 and that the β -functions vanish [5]:

$$\sum_{e: \text{around } f} R_e = 2, \quad \sum_{e: \text{incident to } v} (1 - R_e) = 2, \quad (5)$$

for all $f \in F, v \in V$. In these expressions we used an elliptic gamma function

$$\Gamma(x; p, q) = \prod_{j, k \geq 0} \frac{1 - x^{-1} p^j q^k}{1 - x p^j q^k}, \quad (6)$$

and $\kappa := \prod_{j \geq 0} (1 - p^{j+1})(1 - q^{j+1})$.

Our key observation is that the index of our 4d theory is identified with the partition function of the Bazhanov-Sergeev spin model [9] (see also [10]) under the following identification: [24]

4d gauge theory	spin model
quiver diagram	spin lattice
Cartan variable	spin variable
1-loop determinant	Boltzmann weight
R-charge	spectral parameter
4d index	partition function

In this correspondence, the invariance of the 4d index under Seiberg duality [11] results from the star-triangle relation in the Bazhanov-Sergeev spin model [12]. This spin system can be reduced to many known integrable models, for example the chiral Potts model [9]. It is known that the partition function of the chiral Potts model gives a special value of Jones polynomial and HOMFLY (Hoste-Oceanu-Millett-Freyd-Lickorish-Yetter) polynomial for a link whose projection gives the zig-zag paths. This suggests that the reduction process, which is different from the dimensional reduction discussed in this Letter, connects the 4d quiver gauge theory to the 3d Chern-Simons theory on the link complement. It would be interesting to give a gauge theory interpretation of this novel reduction.

Our 4d gauge theory has a brane realization in terms of N D5-branes and an NS5-brane [13]; in the notation of introduction, we have $D = 6, d_1 = 2, d_2 = 4, \mathcal{C} = T^2$ and $\mathcal{S} = S^3 \times S^1$ [25]. However, it should be kept in mind that our correspondence has important differences from [3] and their variants. For example, our 2d spin system is a *classical* spin system, whereas in [3] the 2d system is the quantum Liouville theory.

3d versus 3d.— We now consider a dimensional reduction of our theory; we take the radius of thermal S^1 to zero, and all the KK modes decouple. In this limit, all the chemical potentials go to 1, but we can take the limit while keeping their ratio finite. Our limit is $\beta \rightarrow 0$ with [14]

$$p = e^{-\beta(1+\eta)}, \quad q = e^{-\beta(1-\eta)}, \quad (7)$$

after which 4d $\mathcal{N} = 1$ theory on $S^3 \times S^1$ reduces to 3d $\mathcal{N} = 2$ theory on an ellipsoid S_b^3 [15], where $b^2 = \frac{1+\eta}{1-\eta}$. In addition we Higgs the theory to the Cartan, by giving a

VEV (Vacuum Expectation Value) to the vectormultiplet scalar of the diagonal gauge group $U(1)_{\text{diag}} \subset SU(2)^{|V|}$ and sending it to infinity. The 4d index now reduces to

$$Z_{3d}(R) = \int \prod_{v \in V} d\sigma_v \prod_{e \in E} Z_{\text{hyper}}^e(\sigma; R), \quad (8)$$

where

$$Z_{\text{hyper}}^e(\sigma; R) = \frac{s_b \left(\sigma_{s(e)} - \sigma_{t(e)} + \frac{iQ}{2}(1 - R_e) \right)}{s_b \left(\sigma_{s(e)} - \sigma_{t(e)} - \frac{iQ}{2}(1 - R_e) \right)}, \quad (9)$$

the variable σ_v is the vectormultiplet scalar, and $s_b(x)$ is the quantum dilogarithm function. The integral in (8) is taken over the real axis. The deformation parameter b plays the role of the quantum parameter of the theory. The result (8) coincides with the partition function of the Faddeev-Volkov model [16, 17], which describes a discrete Virasoro symmetry.

In the semiclassical limit $b \rightarrow 0$, S_b^3 reduces to $\mathbb{R}^2 \times S_b^1$ with S_b^1 of small radius b , and the theory effectively reduces to a 2d theory with all the Kaluza-Klein modes included. Indeed, (after rescaling σ) the 3d partition function reduces to an integral of the effective twisted superpotential $\mathcal{W}_{2d}(\sigma; R)$:

$$Z_{3d}(R) = \int \prod_{v \in V} d\sigma_v \exp \left[\frac{1}{2\pi b^2} \mathcal{W}_{2d}(\sigma) \right], \quad (10)$$

where

$$\mathcal{W}_{2d}(\sigma; R) = \sum_{e \in E} \left[l(\sigma_{s(e)} - \sigma_{t(e)} + i\theta_e^*) - l(\sigma_{s(e)} - \sigma_{t(e)} - i\theta_e^*) \right]. \quad (11)$$

Here we defined $\theta_e^* = \pi(1 - R_e)$ and $l(z)$ is defined from the classical dilogarithm function $\text{Li}_2(z)$ to be

$$l(z) = \text{Li}_2(-e^z) + \frac{1}{4}z^2. \quad (12)$$

An important result in [17, 18] states that the saddle point equation of the effective twisted superpotential can be interpreted as a gluing condition at the vertices of the quiver diagram of (non-ideal) hyperbolic tetrahedra whose projection to $\partial\mathbb{H}^3$ is a triangle in Figure 2.

The necessary and sufficient conditions for the existence of the solution of the gluing condition is stated in [18, Theorem 3]. The first condition is (in our notation)

$$\sum_{f \in F} 2\pi = \sum_{e \in E} 2(\pi - \theta_e^*). \quad (13)$$

The second condition is that for a nonempty subset F' of F with $F \neq F'$ and the set E' of all edges incident with any face of F' , we have

$$\sum_{f \in F'} 2\pi < \sum_{e \in E'} 2(\pi - \theta_e^*). \quad (14)$$

These conditions follow from the conditions on the R-charge (5); the first (second) condition follows from the sum of the first equation of (5) over $f \in F$ ($f \in F'$). Note that each edge is adjacent to two faces.

After gluing tetrahedra we therefore have a 3d hyperbolic manifold M_R whose projection to $\partial\mathbb{H}^3$ is combinatorially given by the bipartite graph on our T^2 . When the circle radii are all of the same value, the Legendre transform of the volume of M_R with respect to the angles θ_e^* is related with the prepotential of the topological string theory on the dual toric Calabi-Yau 3-fold [4].

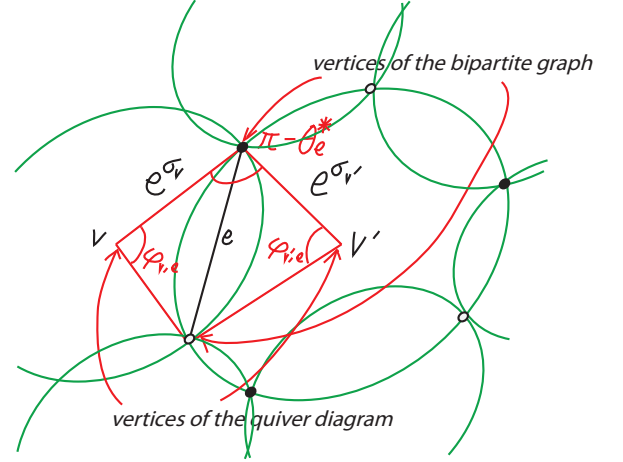


FIG. 2: The projection of a non-ideal hyperbolic tetrahedron gives the triangle in the Figure. Our 3-manifold M_R is obtained by gluing these tetrahedra.

The correspondence between 2d twisted superpotential and the 3d classical hyperbolic geometry motivates us to propose a quantum version of the correspondence: the 3d partition function Z_{3d} with finite b compute the partition function of the 3d $SL(2)$ Chern-Simons theory, where b is related to the level t by $t \sim 1/b^2$. Since 3d gravity is closely related with 3d $SL(2)$ Chern-Simons theory [26], this means that our 2d spin system is a version of the spin network for 3d gravity [27], i.e. a spin system defined on a discrete lattice which reproduces 3d gravity in a limit.

Finally, the results of this Letter are reminiscent of two existing results in the literature. The first is the 4d/2d relation between the 4d superconformal index for Gaiotto theories and the correlation function of 2d TQFT (Topological Quantum Field Theory) [19]. The second is the 3d/3d relation between 3d $\mathcal{N} = 2$ theories and 3d $SL(2)$ Chern-Simons theories [20]. It would be interesting to elucidate the precise relation between these and the results of this Letter.

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- [21] The quantum geometry is captured by an $SL(2)$ Chern-Simons theory. The role of the Planck constant is played by the inverse of the level t of the Chern-Simons theory, and the classical geometry is reproduced in the saddle point approximation of the limit $t \rightarrow \infty$. Fluctuations around the saddle point gives a number of interesting enumerative invariants for 3-manifolds, for example the Reidemeister-Ray-Singer torsion.
- [22] The number of dimensions decreases in dimensional reduction. Dimensional oxidation is the opposite, where the number of dimensions increases in the process.
- [23] We can also include flavor chemical potentials, and this will shift the R-charge when dimensionally reduced to 3d.
- [24] There are in fact differences between the two. First, our model is defined on T^2 , on the other hand their model is defined on \mathbb{R}^2 . Second, we do not include a spin-independent normalization factor $\kappa(\alpha)$ in the weight of Bazhanov-Sergeev spin model. This only changes the overall normalization of the partition function (*i.e.* only the normalization outside the integral of (2)), and our index is still invariant under the double Yang-Baxter move [4].
- [25] The topology of \mathcal{C} is unique, however there is an extra ingredient, an NS5-brane, which wraps a general Riemann surface.
- [26] There are important differences between the two. However, the difference is irrelevant for the consideration of this paper since we only consider saddle point expansions around geometric flat connections.
- [27] Our spins take continuous values, and transforms as a continuous representation of $SL(2)$. This is in contrast with many literature on spin networks, where the spins are the discrete spins under the compact group $SU(2)$, not the non-compact $SL(2)$ group as it really expected from 3d gravity.